

# Optico-Mechanical Analogy: an Axiomatic Approach

Alexander Afriat

Published online: 8 May 2007  
© Springer Science+Business Media, LLC 2007

**Abstract** An axiomatic characterization of a ‘two-level Hamiltonian structure’ is proposed, which expresses the optico-mechanical analogy by representing optics and mechanics as (disjoint) classes of models satisfying the axioms. There is the ‘Hamilton–Jacobi level’, which involves a differential manifold on which the characteristic function satisfying the Hamilton–Jacobi equation is defined; and the ‘symplectic level’, involving the Hamiltonian, defined on the cotangent bundle of the manifold. The two levels, with the (analogous) structures on them, concern both optics and mechanics.

## 1 Introduction

The optico-mechanical analogy, which arguably had a central role in the development of wave mechanics,<sup>1</sup> is usually expressed<sup>2</sup> by describing analytical mechanics and geometrical optics separately, then pointing out a handful of similarities, sometimes by means of a ‘dictionary’ indicating correspondences between analogous objects: between the mechanical potential and the optical index of refraction, between mechanical trajectories and optical rays and so forth. In such approaches the ‘analogy’ has a tenuous, insubstantial status, entirely subordinate to the theories, which are primary. To give the analogy itself more substance and an appropriate primacy, to turn it into an object in its own right (and “give to airy nothing a local habitation and a name”), one can, rather than just noting a few resemblances, abstract a common structure<sup>3</sup> possessing optical and mechanical models. The difference is perhaps one of emphasis and degree, for even a statement of similarities might be viewed as a rough description of structure; but the degree seems considerable.

---

A. Afriat (✉)  
Istituto di Filosofia, Università di Urbino, Via Timoteo Viti 10, 61029 Urbino, Italy  
e-mail: afriat@gmail.com

<sup>1</sup> See [3] and [7, 8]; but also [14].

<sup>2</sup> See [1] for instance.

<sup>3</sup> “Structure”, in the language of model theory, is often a synonym of “model”. Here it is more like a synonym of “theory”.

Of the various resources available for the characterization of structure, those of set-theoretical axiomatization, associated chiefly with Patrick Suppes, seem appropriate here and will be adopted. The axioms will not be intended—unlike those of, say, a group or a vector space—as a ‘necessary and sufficient’ basis for deduction, only as a summary of structure, a delineation of *Gestalt*. An appropriate ‘completeness’ (perhaps allowing the derivation of all important theorems) and ‘independence’ are sometimes expected of axioms, but will not be sought here; genuine completeness is unattainable, practically at any rate, and independence is both hard to characterize exactly and unnecessarily stringent. The axioms are only meant to emphasize certain features of the structure that best represent it and are judged significant.

## 2 Set-Theoretical Axiomatization

Set-theoretical methods have been explicitly used for the axiomatization of scientific theories since McKinsey et al. [4] did so, giving rise to a considerable ‘structuralist’<sup>4</sup> literature. Patrick Suppes, who sought to extend the methods of Bourbaki to the natural sciences, has had a fundamental role in the programme, especially in its early days; subsequent developments, stressing semantics, have also been due to Sneed [9], Stegmüller [10, 11], Balzer et al. [2] and others. Mechanical theories have served as paradigmatic examples since the beginning: ‘classical particle mechanics’ was axiomatized in [4, 5, 12, 13]; Lagrangian and Hamiltonian mechanics in [9]. Optical theories have received less attention. Set-theoretical characterization of Hamilton–Jacobi theory or the optico-mechanical analogy has not yet been attempted.

*Economy* is central to the approach; set-theoretical methods lend themselves to the omission of what is taken for granted or ‘understood,’ such as rules of inference, set theory itself, or staple mathematics. No attempt is made at exhaustive characterization of logical structure; only what is considered relevant and worth mentioning is included. Even the simplest scientific theories can involve many sets, functions and axioms, many more than deserve attention; most are left out.

Suppose we want to give a standard formalization of elementary probability theory. On the one hand, if we follow the standard approach, we need axioms about sets to make sense even of talk about the joint occurrence of two events, for their events are represented as sets and their joint occurrence as their set-theoretical intersection. On the other hand, we need axioms about the real numbers as well, for we shall want to talk about the numerical probabilities of events. Finally, after stating a group of axioms on sets, and another group on the real numbers, we are in a position to state the axioms that belong just to probability theory as it is usually conceived. In this welter of axioms, those proper to probability theory can easily be lost sight of. More important, it is senseless and uninteresting continually to repeat these general axioms on sets and on numbers whenever we consider formalizing a scientific theory. No one does it, and for good reason.<sup>5</sup>

<sup>4</sup>“Sneed, auf den die hier vorgetragene ... Auffassung zurückgeht, hatte sein methodisches Vorgehen ursprünglich als ‘non-statement view’ bezeichnet. ... Bar-Hillel ... hatte ... den Vorschlag gemacht, diese unshöne negative Kennzeichnung durch eine positive zu ersetzen und dafür die Bezeichnung “strukturalistische Auffassung von Theorien”, kurz: “Strukturalismus”, zu verwenden...” [11, p. 2].

<sup>5</sup>[13, p. 27].

A theory is characterized in the structuralist approach by a set-theoretical predicate—like ‘is a classical particle mechanics’ or ‘is a group’—involving the basic objects and axioms of the theory; or equivalently by the extension of the predicate, in other words by the class of models satisfying the axioms.

Here a theory with very different models is axiomatized. Set-theoretical methods free the mathematical theory from association with particular kinds of models, and can be applied, unlike other formal methods, without regard for specific application.<sup>6</sup> Furthermore a given set-theoretical structure can not only admit diverse physical models, but even abstract mathematical models:

It is one of the theses of this book that there is no theoretical way of drawing a sharp distinction between a piece of pure mathematics and a piece of theoretical science. The set-theoretical definitions of the theory of mechanics, the theory of thermodynamics, and a theory of learning, to give three rather disparate examples, are on all fours with the definitions of the purely mathematical theories of groups, rings, fields, etc. From a philosophical standpoint there is no sharp distinction between pure and applied mathematics, in spite of much talk to the contrary.<sup>7</sup>

Indeed it will be unclear, and unimportant, whether mathematics or physics is really at issue here.

What is meant here by ‘optico-mechanical analogy’ is that two disjoint classes of models—optical and mechanical—satisfy the axioms proposed. The optical models only add semantics to the abstract scheme, whereas further logical structure (basic objects and axioms) is provided for mechanical models.

The general outlines of the structure common to analytical mechanics and geometrical optics will be described with broad brushstrokes, without aiming at generality or rigor. Coordinates will be avoided to keep the presentation simple and clean, not for rigor. Global issues—conjugate points etc.—will also be avoided, however relevant they may be to optics.

### 3 Two-Level Hamiltonian Structure

The theory expressing the optico-mechanical analogy will itself involve a further analogy, between two ‘substructures’ it contains, one on an underlying manifold, the other on its cotangent bundle. At either level, briefly, the differential of a real-valued function (the characteristic function on the underlying manifold, the Hamiltonian on the cotangent bundle) gets turned into a vector field tangent to a congruence by means of a twice-covariant tensor (the metric tensor in one case, the symplectic two-form in the other); one can say the function *generates* the congruence. Of course there are differences: the symplectic two-form is skew-symmetric whereas the metric tensor is symmetric; the Hamiltonian is largely arbitrary, whereas the characteristic function has to satisfy a differential equation (but with much freedom in the choice of initial conditions). One can speak, perhaps loosely, of a ‘Hamilton–Jacobi’ substructure on the underlying manifold and of a ‘symplectic’ substructure on the cotangent bundle. Together they make a ‘two-level Hamiltonian structure.’

<sup>6</sup>“... the Suppes approach ... frees us from the prejudice (implicit in the formal language approach) that the formalized theory must have just one big intended interpretation ...” [6, p. 74].

<sup>7</sup>[13, p. 33].

Both substructures involve the six objects  $M, f_M, df_M, \omega_M^\#, X_M, \Phi_M$ : a manifold  $M$ , a real-valued function  $f_M$  defined on it, the differential  $df_M$ , and a twice-covariant tensor  $\omega_M^\#$  used to turn  $df_M$  into the vector field  $X_M = (df_M)^\# = \omega_M^\#(df_M)$  tangent to the congruence  $\Phi_M$ . The generic manifold or ‘index’  $M$  can have two values,  $Q$  and  $\Gamma$ , where

$$\Gamma = T^*Q = \bigcup_{q \in Q} T_q^*Q$$

is the cotangent bundle of the manifold  $Q$ , and  $T_q^*Q$  the cotangent space dual to the space  $T_qQ$  tangent at  $q$  to  $Q$ . For  $M = \Gamma$  we have the symplectic level, where  $f_\Gamma$  is the Hamiltonian  $H : \Gamma \rightarrow \mathbb{R}$  and  $\omega_\Gamma^\#$  the symplectic two-form  $\Omega^\#$  (naturally induced by the cotangent bundle structure), which at a point  $z$  of  $\Gamma$  is the transformation  $\Omega^\#(z) : T_z^*\Gamma \rightarrow T_z\Gamma$  from the cotangent space  $T_z^*\Gamma$  to the tangent space  $T_z\Gamma$ . Setting  $M = Q$  gives the Hamilton–Jacobi level, where  $f_Q$  is the characteristic function  $W : Q \rightarrow \mathbb{R}$ ,  $\omega_Q^\#$  the metric tensor  $g^\# : T_q^*Q \rightarrow T_qQ$  (all  $q \in Q$ ),  $df_Q$  the covector  $p = dW \in T_q^*Q$ , and  $X_Q$  the vector  $\dot{q} = p^\# = g^\#(dW) \in T_qQ$ .

The two substructures ‘communicate.’ The projection  $\pi(\Phi_\Gamma)$  of the congruence  $\Phi_\Gamma$  onto the underlying manifold  $Q$  is a crisscrossing tangle, not another congruence. Hamilton–Jacobi theory will be viewed here as a method of extracting a congruence  $\Phi_Q$  on  $Q$  out of the tangle by means of an arbitrary ‘initial’ surface  $Q_0 \subset Q$  (of codimension one in  $Q$ ), together with the *Hamilton–Jacobi equation*  $H(q, dW) = \text{constant}$ —which provides a further connection between the two levels, by relating a function  $H = f_\Gamma$  on  $\Gamma$  with the differential of a function  $W = f_Q$  on  $Q$ . Physically  $\Phi_\Gamma$  will represent the ‘fictitious’ set of all possibilities—obtained varying over *all* initial conditions—compatible with the Hamiltonian; and  $\Phi_Q$  the smaller set of all possibilities compatible with the Hamiltonian and with a particular class (selected by  $Q_0$ ) of initial conditions.

The two substructures form the core of the two-level Hamiltonian structure. Of the aforementioned six objects we will choose three as basic: the function  $f_M$  (as the ‘generator’ or ‘input’), the twice-covariant tensor  $\omega_M^\#$  (as the fundamental transformation), and the congruence  $\Phi_M$  (as the ‘output’)—the differential  $df_M$  and vector field  $X_M$  being too closely related to  $f_M$  and  $\Phi_M$  to be worth including. Rather than the index  $M$  we will take its two values,  $Q$  and  $\Gamma$ , and also the initial surface  $Q_0 \subset Q$ . Though basic, the objects  $\mathfrak{H} = (Q, \Gamma, f_M, \omega_M^\#, \Phi_M, Q_0)$  are not independent; the three freely-chosen ones are the manifold  $Q$ , which determines  $\Gamma$ ; the function  $f_\Gamma$ , which then generates the congruence  $\Phi_\Gamma$ ; and finally the surface  $Q_0$ , which gives  $f_Q$  and  $\Phi_Q$ .

As axioms we adopt:

**Axiom 1**  $\Gamma = T^*Q$  is the cotangent bundle of the  $n$ -dimensional differential manifold  $Q$ .

**Axiom 2**  $f_M : M \rightarrow \mathbb{R}$ , where  $M = Q, \Gamma$ .

**Axiom 3** The vector field  $\omega_M^\#(df_M)$  is tangent to the congruence  $\Phi_M$  on  $M$ , where  $\omega_M^\#$  is a twice-covariant tensor field,  $\mu \mapsto \omega_M^\#(\mu) : T_\mu^*M \rightarrow T_\mu M, \forall \mu \in M$ .

**Axiom 4** The function  $f_Q : Q \rightarrow \mathbb{R} : q \mapsto f_Q(q)$  with  $(n - 1)$ -dimensional level surface  $Q_0 \subset Q$  satisfies equation  $f_\Gamma(q, df_Q) = \text{constant}$ .

The extension of the predicate ‘is a two-level Hamiltonian structure’ is the class of all systems  $\mathfrak{H}$  satisfying Axioms 1–4.

A few words on how the initial surface  $Q_0$  and condition  $H(q, dW) = \text{constant}$  determine  $W$ , under the (typically mechanical) simplifying assumption that the Hamiltonian is of the form  $H = \langle p, p^\# \rangle / 2 + V(q)$ , where  $\langle x, y \rangle$  is the value of the covector  $x$  at the vector  $y$ . A distance and a direction will tell us how to advance from a point  $q_0$  of  $Q_0$ : the distance is provided by the condition  $\langle p, p^\# \rangle / 2 + V(q_0) = \text{constant}$ , the direction by  $Q_0$ , in fact by the  $(n - 1)$ -dimensional space  $T_{q_0} Q_0 \subset T_{q_0} Q$  tangent to  $Q_0$  at  $q_0$ . Calling the constant  $\eta$ , condition  $\langle p, p^\# \rangle = 2[\eta - V(q_0)]$  determines a sphere

$$\Sigma_{q_0}^\eta = \{p \in T_{q_0}^* Q_0 : \langle p, p^\# \rangle = 2[\eta - V(q_0)]\}$$

of radius  $\sqrt{2[\eta - V(q_0)]}$ , which is the required distance. The hyperplane  $T_{q_0} Q_0$  determines a direction  $\alpha_0 = (T_{q_0} Q_0)^\perp$  ‘normal’ to it, where

$$(T_{q_0} Q_0)^\perp = \{p \in T_{q_0}^* Q : \langle p, \dot{q} \rangle = 0, \dot{q} \in T_{q_0} Q_0\}$$

is the annihilator of  $T_{q_0} Q_0$ ;  $\dim \alpha_0 = \dim T_{q_0} Q - \dim T_{q_0} Q_0 = 1$ . Of the two points in  $\alpha_0 \cap \Sigma_{q_0}^\eta$ , we choose the one, say  $p_0$ , that lies in the direction we want. And a momentum  $p_0 = dW_0$  for every  $q_0 \in Q_0$  is enough, together with the congruence  $\Phi_\Gamma$ , to determine the differential  $dW$ , the vector field  $\dot{q} = g^\#(dW)$  tangent to  $\Phi_Q$ , and  $W$  up to an additive constant. For the unique curve  $\gamma_\Gamma : [a, b] \rightarrow \Gamma : t \mapsto (q(t), p(t))$  of  $\Phi_\Gamma$  passing through  $z_0 = (q_0, dW_0) \in \Gamma$  assigns a differential  $dW(t) = p(t)$  to every point  $q(t) = \gamma_Q(t)$  in the image  $\bar{\gamma}_Q \subset Q$  of the projection  $\pi(\gamma_\Gamma) = \gamma_Q : [a, b] \rightarrow Q$  of  $\gamma_\Gamma$  onto  $Q$ . Doing the same for the rest of  $Q_0$  we cover  $Q$  and obtain  $\Phi_Q$ .

### 4 Optics

In an optical model,  $Q$  will typically be a three-dimensional Euclidean *luminiferous medium*  $\mathbb{E}$ , whose properties are characterized by the Hamiltonian  $H = f_\Gamma$ . If the restriction  $H_q = H|_{T_q^* \mathbb{E}} : T_q^* \mathbb{E} \rightarrow \mathbb{R}$  is the same (for instance, the same constant) for every point  $q$  of  $\mathbb{E}$  the medium is *homogeneous*, otherwise *inhomogeneous*; if  $H_q$  has spherical symmetry (with respect to the background Euclidean metric) on every cotangent space  $T_q^* \mathbb{E}$  the medium is *isotropic*, otherwise *anisotropic*. Optical anisotropies tend to be of the ‘ellipsoidal’ kind represented by a tensor (field); the Hamiltonian will then be of the form  $H = \langle p, g^\#(q)(p) \rangle$ , where the tensor  $g^\#(q)$  varies over the medium. But inhomogeneity and anisotropy are in principle independent, and can in some cases be neatly separated; when the same ellipsoidal anisotropy is common to the whole medium one can write  $H = \langle p, \bar{g}^\#(p) \rangle + V(q)$ , where the inhomogeneity is expressed by  $V(q)$  and the common anisotropy by the constant tensor  $\bar{g}^\#$ . ‘Isotropies’ are represented by degeneracies (equal eigenvalues) of the tensor  $g^\#$  (or  $\bar{g}^\#$ ); if  $g^\#$  is a multiple  $g^\# = g e^\#$  of the background Euclidean metric  $e^\#$  there will be complete isotropy and one can write  $H = \bar{g} \langle p, e^\#(p) \rangle + V(q)$ , where  $\bar{g}$  is a constant scalar.

The characteristic function  $W = f_Q$  represents *time elapsed*, from when the luminous disturbance leaves the fundamental level surface  $Q_0$ , here a *light source*. The level surfaces of  $W$  are *surfaces of equal time*. A point  $(q, p)$  on the cotangent bundle  $\Gamma$  indicates, in addition to a *position*  $q \in Q$ , a *normal slowness* (to use Hamilton’s term)  $p = dW$ , whose modulus is large where the surfaces of equal time are close together, small where they are far apart. So  $\Gamma$  is the set of all positions together with all normal slownesses. The modulus of the ‘contravariant slowness’ or ‘inverse velocity’ (neither term being Hamilton’s)  $\dot{q} = p^\# = dH_q(p)$  is also large where the surfaces are close together. The congruence  $\Phi_\Gamma$  represents the virtual set of all rays compatible with the properties of the medium,  $\Phi_Q$  the more real set of all rays emanating orthogonally from the source  $Q_0$ .

### 5 Mechanics

In mechanical models  $Q$  is a *configuration space* (determined by ideal holonomic scleronomous constraints), the cotangent bundle  $\Gamma$  *phase space*. A point  $(q, p) \in \Gamma$  gives a *position*  $q$  together with a *momentum*  $p$ . The characteristic function  $W$  represents *action*, its level surfaces (such as  $Q_0$ ) *surfaces of equal action*. The congruence  $\Phi_\Gamma$  is the virtual set of all dynamical trajectories, for all possible initial conditions, generated by the Hamiltonian  $H$ , which represents *energy*. The smaller congruence  $\Phi_Q$  is the set of trajectories crossing the level surface  $Q_0$  orthogonally.

This remains rather abstract and general. We can restrict the models to a typical class involving point masses ('PM') by adding the basic objects  $P, m, V$  and axioms:

**Axiom 5**  $P = \{p_1, \dots, p_N\}$ .

**Axiom 6**  $m : P \rightarrow \mathbb{R}^+ : p_r \mapsto m_r = m(p_r)$ .

**Axiom 7**  $V : Q \rightarrow \mathbb{R}$ .

**Axiom 8**  $H(q, p) = \frac{1}{2} \langle p, g^\#(m_1, \dots, m_N)(p) \rangle + V$ .

A system  $\mathfrak{H}' = (Q, \Gamma, f_M, \omega_M^\#, \Phi_M, Q_0, P, m, V)$  satisfying Axioms 1–8 will be a 'PM model' of the two-level Hamiltonian structure.

The mapping  $m$  assigns a positive and finite real number  $m_r = m(p_r)$  to each element  $p_r$  of the finite nonempty set  $P$ . How the values  $m_1, \dots, m_N$  figure in the Hamiltonian requires comment. We begin with the empty three-dimensional Euclidean space  $\mathbb{E}$ , whose geometry is characterized by the Euclidean metric  $e^\# : T_q^* \mathbb{E} \rightarrow T_q \mathbb{E}$  (all  $q \in \mathbb{E}$ ). The metric  $e_N^\# : T_q^* \mathbb{E}^N \rightarrow T_q \mathbb{E}^N$  of the  $N$ -th Cartesian power  $\mathbb{E}^N = \mathbb{E} \times \dots \times \mathbb{E}$  remains Euclidean. The non-Euclidean manifold  $Q \subset \mathbb{E}^N$  is an  $n$ -dimensional ( $n \leq 3N$ ) subset of  $\mathbb{E}^N$ , and  $T_q Q \subset T_q \mathbb{E}$  for all  $q \in Q$ . The non-Euclidean metric  $e_Q^\# : T_q^* Q \rightarrow T_q Q$  agrees with  $e_N^\#$  on all of the smaller domain  $T_q^* Q$ ; it is the restriction  $e_Q^\# = e_N^\#|_{T_q^* Q}$  of  $e_N^\#$  to  $T_q^* Q$ .

To introduce  $m_1, \dots, m_N$  we first break up each cotangent space  $T_q^* Q$  into  $N$  mutually orthogonal subspaces  $T_q^* Q_1, \dots, T_q^* Q_N$ , where  $1 \leq \dim T_q^* Q_r \leq 3$ , and

$$T_q^* Q = \bigoplus_{r=1}^N T_q^* Q_r.$$

The projection with range  $T_q^* Q_r$  is denoted  $\Pi_r : T_q^* Q \rightarrow T_q^* Q_r$ . We will need the composition  $\Pi_r^\# = e_Q^\# \circ \Pi_r : T_q^* Q \rightarrow T_q Q_r$ , which 'projects' onto the subspace  $T_q Q_r$ .

So each  $p_r \in P$  now has its own  $m_r$ , as well as a one-, two- or three-dimensional subspace  $T_q^* Q_r \subset T_q^* Q$ , and a  $\Pi_r^\#$ . Writing

$$g^\# = \sum_{r=1}^N m_r^{-1} \Pi_r^\#,$$

we have a metric which associates an  $m_r^{-1}$  with every subspace  $T_q Q_r$ ; and the scalar product  $\langle p, g^\#(p) \rangle$ , central to the Hamiltonian, is based on  $g^\#$ . This explains how the values  $m_r = m(p_r)$  assigned to the elements  $p_r \in P$  enter into the Hamiltonian.

The elements of  $P$  are of course *particles*. The function  $m$  represents *mass*, and  $V$  *potential energy*—which, in the spirit of analytical mechanics, is included instead of *force*,  $dV$ .

## 6 Final remarks

I have associated terms like ‘energy’ or ‘light source’ with the basic objects and left it at that, deliberately avoiding the separate and delicate issues of what such things really are and how they ought to be measured. McKinsey et al. [4] expressed a similar attitude:

Thus we leave aside all problems connected with the measurement of time (as well as of mass, distance, and force); this is not because we believe all such problems have been solved—or that they are unimportant—but merely because we consider it is possible to separate mechanics proper from such epistemological and experimental questions.

They continued in a footnote:

We may add that the attempt to deal with both sorts of problems simultaneously is in our opinion responsible for much of the confusion and murkiness characteristic of the usual discussions of the foundations of physics.

Truesdell replied with the following footnote in the same article:

The communicator is in complete disagreement with the view of classical mechanics expressed in this article. . . . While he does not believe the present work achieves any progress whatever toward the precision of the concept of force, which always has been and remains still the central conceptual problem, and indeed the only one not essentially trivial, in the foundations of classical mechanics. . . .

The focus here has been, rather, on structural analogy and models.

**Acknowledgements** I wish to thank Jeremy Butterfield, Sebastiano Carpi, Dennis Dieks, Hans Duistermaat, Remko Muis, Fred Muller, Michael Seevinck and Jos Uffink for many fruitful discussions.

## References

1. Arnold, V.I.: *Mathematical Methods of Classical Mechanics*. Springer, Berlin (1997)
2. Balzer, W., Moulines, C.U., Sneed, J.D.: *An Architectonic for Science*. Kluwer, Dordrecht (1987)
3. de Broglie, L.: *Recherches sur la théorie des quanta*. *Ann. Phys.* **3**, 22–128 (1925)
4. McKinsey, J.C.C., Sugar, A.C., Suppes, P.: *Axiomatic foundations of classical particle mechanics*. *J. Ration. Mech. Anal.* **2**, 253–72 (1953)
5. McKinsey, J.C.C., Suppes, P.: *Transformations of systems of classical particle mechanics*. *J. Ration. Mech. Anal.* **2**, 273–89 (1953)
6. Moulines, U.M., Sneed, J.D.: *Suppes’ philosophy of physics*. In: Bogdan, R. (ed.) *Patrick Suppes*. pp. 59–91. Reidel, Dordrecht (1979)
7. Schrödinger, E.: *Quantisierung als Eigenwertproblem (zweite Mitteilung)*. *Ann. Phys.* **79**, 489–527 (1926)
8. Schrödinger, E.: *Der Grundgedanke der Wellenmechanik*. In: Schrödinger, E. (ed.) *Was ist ein Naturgesetz?*. Oldenbourg, Munich (1987)
9. Sneed, J.D.: *The Logical Structure of Mathematical Physics*. Kluwer, Dordrecht (1979)
10. Stegmüller, W.: *Theorie und Erfahrung*. Springer, Berlin (1973)
11. Stegmüller, W.: *Neue Wege der Wissenschaftsphilosophie*. Springer, Berlin (1980)
12. Suppes, P.: *Introduction to Logic*. Dover, New York (1999)
13. Suppes, P.: *Representation and Invariance of Scientific Structures*. CSLI, Stanford (2002)
14. Wessel, L.: *Schrödinger’s route to wave mechanics*. *Stud. Hist. Philos. Sci.* **10**, 311–40 (1979)